Derk Rouwhorst

S2117398

$9\frac{1}{2}/9\frac{1}{2}/9/8\frac{1}{2} = 36\frac{1}{2}$

9

1 $H(\lambda)\psi(\lambda) = E(\lambda)\psi(\lambda)$ $\quad 2/2/1\frac{1}{2}/4$

~~Use Ehrenfest~~ $\left\langle \frac{\partial H}{\partial \lambda} \right\rangle = \left\langle \psi(\lambda) \middle| \frac{\partial H}{\partial \lambda} \middle| \psi(\lambda) \right\rangle$

$$= \left\langle \psi(\lambda) \middle| \frac{\partial}{\partial \lambda}\left(H(\lambda)\psi(\lambda)\right) \right\rangle = \left\langle \psi(\lambda) \middle| \frac{\partial}{\partial \lambda}\left(E(\lambda)\psi(\lambda)\right) \right\rangle$$

$$= \left\langle \psi(\lambda) \middle| \frac{\partial E}{\partial \lambda}\psi(\lambda) + E(\lambda)\frac{\partial \psi}{\partial \lambda} \right\rangle = \frac{\partial E}{\partial \lambda}\langle \psi(\lambda)|\psi(\lambda)\rangle + \left\langle \psi(\lambda) \middle| E(\lambda)\frac{\partial \psi}{\partial \lambda} \right\rangle$$

$$= \frac{\partial E}{\partial \lambda} + E(\lambda)\left\langle \psi(\lambda) \middle| \frac{\partial \psi}{\partial \lambda} \right\rangle = \frac{\partial E}{\partial \lambda}$$

$$H = \frac{p_r^2}{2m} + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2} - Z\frac{a_0}{r}\alpha^2 mc^2, \qquad p_r = -i\hbar\frac{1}{r}\frac{\partial}{\partial r} r$$

$$E_n = -\frac{1}{2}\left(\frac{Z\alpha}{n}\right)^2 mc^2, \quad n = n_r + l + 1, \quad a_0 = \frac{\hbar}{\alpha mc}$$

(b) $\left\langle \frac{\partial H}{\partial Z} \right\rangle = -\left\langle \frac{a_0}{r}\alpha^2 mc^2 \right\rangle = -a_0\alpha^2 mc^2 \left\langle \frac{1}{r} \right\rangle$

$$\frac{\partial E}{\partial Z} = -\left(\frac{\alpha}{n}\right)^2 Z mc^2$$

$$\Rightarrow \left\langle \frac{1}{r} \right\rangle = \frac{\left(\frac{\alpha}{n}\right)^2 Z mc^2}{a_0\alpha^2 mc^2} = \frac{Z}{a_0 n^2}$$

(c) $\left\langle \frac{\partial H}{\partial l} \right\rangle = \left\langle \frac{\hbar^2}{2m}\frac{2l+1}{r^2} \right\rangle = \frac{(2l+1)\hbar^2}{2m}\left\langle \frac{1}{r^2} \right\rangle$

$$\frac{\partial E}{\partial l} = \frac{\partial E}{\partial n}\frac{\partial n}{\partial l} = (Z\alpha)^2 n^{-3} mc^2$$

$$\Rightarrow \left\langle \frac{1}{r^2} \right\rangle = \frac{(Z\alpha)^2 n^{-3} mc^2}{\frac{(2l+1)\hbar^2}{2m}} = \frac{2(Z\alpha)^2 m^2 c^2}{(2l+1)n^3\hbar^2} = \frac{2Z^2}{(2l+1)a_0^2 n^3}$$

1 (d) (i) $[p_r, r^n] = p_r r^n - r^n p_r = -i\hbar \frac{1}{r}\frac{\partial}{\partial r} r^{n+1} - r^n \cdot -i\hbar \frac{1}{r}\frac{\partial}{\partial r} r$

$$= -i\hbar \frac{1}{r}(n+1) r^n + i\hbar r^{n-1} = -(n+1) i\hbar r^{n-1} + i\hbar r^{n-1}$$

$$= -n i\hbar r^{n-1}$$

$$[p_r, H] = \left[p_r, \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right] + \left[p_r, -Z\frac{a_0}{r}\alpha^2 mc^2\right]$$

$$= \frac{\hbar^2}{2m} l(l+1)(2i\hbar r^{-3}) - Z a_0 \alpha^2 mc^2 (i\hbar r^{-2})$$

(ii) $0 = \langle [p_r, H] \rangle = \frac{i\hbar^3}{m} l(l+1) \left\langle \frac{1}{r^3} \right\rangle - i\hbar Z a_0 \alpha^2 mc^2 \left\langle \frac{1}{r^2} \right\rangle$

$$= \frac{i\hbar^3}{m} l(l+1) \cancel{\frac{2Z^2}{(2l+1)a_0 n^3}} \left\langle \frac{1}{r^3} \right\rangle - i\hbar Z a_0 \alpha^2 mc^2 \frac{2Z^2}{(2l+1)a_0 n^3}$$

$$\Rightarrow \left\langle \frac{1}{r^3} \right\rangle = \frac{i\hbar Z \cancel{a_0} \alpha^2 mc^2 \frac{2Z^2}{(2l+1)\cancel{a_0} n^3}}{\frac{i\hbar^3}{m} l(l+1)}$$

$$= \frac{2Z^3 \alpha^2 m^2 c^2}{\hbar^2 (2l+1) l(l+1) n^3} = \frac{2Z^3}{a_0^3 l(l+1)(2l+1) n^3}$$

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3/2/3/$1\frac{1}{2}$

2 (a)

$$H = \begin{bmatrix} E_0 & -A & 0 & 0 \\ -A & E_0 & -A & 0 \\ 0 & -A & E_0 & -A \\ 0 & 0 & -A & E_0 \end{bmatrix}$$

$$|\chi\rangle = \sum_{n=1}^{N} c_n |\phi_n\rangle \qquad \text{"} |\phi_{-1}\rangle := 0, \quad |\phi_{N+1}\rangle := 0 \text{"}$$

$$H|\chi\rangle = \sum_{n=1}^{N} c_n H|\phi_n\rangle \text{"}= \sum_{n=1}^{N} c_n\left(E_0|\phi_n\rangle - A\left(|\phi_{n-1}\rangle + |\phi_{n+1}\rangle\right)\right)\text{"}$$

$$= \sum_{n=0}^{N+1}\left(c_n E_0|\phi_n\rangle - c_{n+1}A|\phi_n\rangle - c_{n-1}A|\phi_n\rangle\right)$$

$$= \sum_{n=0}^{N+1}\left(c_n E_0 - (c_{n+1}+c_{n-1})A\right)|\phi_n\rangle$$

(b) $c_n = c\,(e^{in\delta} - e^{-in\delta})/2i$

$c_0 = c\,(e^{i0\delta} - e^{-i0\delta})/2i = 0$

$0 = c_{N+1} = c\,(e^{i(N+1)\delta} - e^{-i(N+1)\delta})/2i$

$$\Rightarrow e^{i(N+1)\delta} = e^{-i(N+1)\delta}$$

$$\Rightarrow i(N+1)\delta \equiv -i(N+1)\delta \mod 2\pi i$$

$$(N+1)\delta \equiv 0 \mod \pi$$

$$\Rightarrow \delta = \frac{\pi s}{N+1} \qquad s \in \mathbb{N},$$ *for physical relevance:* $s \in \{1, \dots, N\}$

2(c) $H|\phi_n\rangle =$

$$c_{n+1} + c_{n-1} = c\left(e^{i(n+1)\delta} - e^{-i(n+1)\delta} + e^{i(n-1)\delta} - e^{-i(n-1)\delta}\right)/2i$$

$$= c\left(e^{in}\left(e^{i\delta} + e^{-i\delta}\right) - e^{-in}\left(e^{-i\delta} + e^{i\delta}\right)\right)/2i$$

$$= c\,\frac{e^{i\delta} + e^{-i\delta}}{2}\cdot\frac{e^{in} - e^{-in}}{i} = c\cos\delta \cdot 2\sin n$$

$$= 2\cos\left(\frac{\pi S}{N+1}\right) c\sin n$$

$$H|\alpha\rangle = \sum_{n=0}^{N+1}\left(E_0\, c\sin(n\delta) - 2A\cos\left(\frac{\pi S}{N+1}\right) c\sin n\right)|\phi_n\rangle$$

$$\Rightarrow E_\delta = E_0 - 2A\cos\left(\frac{\pi S}{N+1}\right)$$

$$E = E_1 + E_2 + E_3 + E_4 = 4E_0 - 4A\cos\frac{\pi}{5} - 4A\cos\frac{2\pi}{5}$$

$$\simeq 4E_0 - 4A(0.81 + 0.31) = 4E_0 - 4A(1.12)$$

$$= 4(E_0 - A) - 0.48A$$

The effect of delocalization causes the energy to be less then $4(E_0 - A)$; you could view this as the "binding energy" $-0.48A$

(d) $$1 = \sum_{n=1}^{N}|c_n|^2 = \sum_{n=1}^{N}|c|^2\sin^2 n\delta = |c|^2\sum_{n=1}^{N}\sin^2 n\delta$$

$$= |c|^2\,\frac{\pi}{2\delta} = |c|^2\,\frac{N+1}{2}$$

V. $\Rightarrow |c|^2 = \frac{2}{N+1}$, so $c = \pm e^{i\theta}\sqrt{\frac{2}{N+1}}$, or $c = \sqrt{\frac{2}{N+1}}$ as one possibility

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2/3/3/1

3 (a) $T = \frac{l}{v}$, $v = \frac{2\pi\hbar}{M\lambda} = \frac{h}{M\lambda}$ $\Bigg\}$ $T = \frac{Ml\lambda}{2\pi\hbar}$

(b) $H = \frac{\vec{p}^2}{2M} - \mu_n \vec{\sigma} \cdot \vec{B}$

$\vec{B} = (0, 0, B) \Rightarrow \vec{\sigma} \cdot \vec{B} = B\sigma_z \doteq \begin{bmatrix} B & 0 \\ 0 & -B \end{bmatrix}$

$\cancel{\vec{p}^2 = -\hbar^2 \nabla^2}$

$e^{-i\varphi} = e^{-iHT/\hbar} = e^{-i\left(\frac{\vec{p}^2}{2M} - \mu_n \vec{\sigma} \cdot \vec{B}\right) \frac{Ml\lambda}{2\pi\hbar^2}}$

$= \exp\left(-i \frac{\vec{p}^2 l\lambda}{4\pi\hbar^2} I + i \frac{\mu_n M l\lambda B}{2\pi\hbar^2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = \exp\left(\frac{il\lambda}{2\pi\hbar^2}\left(\begin{bmatrix} -p^2 & 0 \\ 0 & -p^2 \end{bmatrix} + \begin{bmatrix} \mu_n MB & \\ & -\mu_n MB \end{bmatrix}\right)\right)$

length $AC = CD = d\cos\theta$

$= \left[e^{-\frac{\vec{p}^2 l\lambda}{8\pi^2\hbar^2} I} \exp\left(\frac{\mu_n M l\lambda}{4\pi^2\hbar^2} \begin{bmatrix} B & \\ & -B \end{bmatrix}\right)\right]^{2\pi i}$

(c) From the result in (b):

$\frac{|\mu_n| M l\lambda (B_2 - B_1)}{2\pi\hbar^2} = 2\pi \Rightarrow \Delta B = B_2 - B_1 = \frac{2\pi}{\frac{|\mu_n| M l\lambda}{2\pi\hbar^2}} = \frac{4\pi^2\hbar^2}{|\mu_n| M l\lambda}$

(d) If you start with no magnetic field and then slowly increase the field strength, then you can observe the maxima in the counting rates. From (c), you can extract $|\mu_n|$, then using (b) gives the phase $e^{-i\varphi}$. Combining this you can prove that under a $2\pi(2k+1)$ rotation a spin $\frac{1}{2}$ neutron changes sign.

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$1\frac{1}{2}/1/3/3$

4 $|\Phi\rangle = (|+_1 \otimes -_2\rangle - |-_1 \otimes +_2\rangle)/\sqrt{2}$

(a) An entangled state is an state with multiple particles (only possible with two particles) where a measurement of the state of one particle determines the state of the other particle.

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$|\Phi\rangle$ is such an entangled state:

measured	state		⇒ particle	state
1	+	$\langle +_1 \otimes \ _2 \vert \Phi\rangle$	⇒ 2	−
1	−	$\langle -_1 \otimes \ _2 \vert \Phi\rangle$	⇒ 2	+
2	+	$\langle \ _1 \otimes +_2 \vert \Phi\rangle$	⇒ 1	−
2	−	$\langle \ _1 \otimes -_2 \vert \Phi\rangle$	⇒ 1	+

$$\mathcal{D}^{(1/2)}(\theta,\phi)\otimes\mathcal{D}^{(1/2)}(\theta,\phi)|\Phi\rangle \doteq \left(\begin{bmatrix} e^{-i\phi/2}\cos\theta/2 \\ e^{i\phi/2}\sin\theta/2 \end{bmatrix}\otimes\begin{bmatrix} -e^{-i\phi/2}\sin\theta/2 \\ e^{i\phi/2}\cos\theta/2 \end{bmatrix} - \begin{bmatrix} -e^{-i\phi/2}\sin\theta/2 \\ e^{i\phi/2}\cos\theta/2 \end{bmatrix}\otimes\begin{bmatrix} e^{-i\phi/2}\cos\theta/2 \\ e^{i\phi/2}\sin\theta/2 \end{bmatrix}\right)/\sqrt{2}$$

$$= \dots = (|+'_1 \otimes -'_2\rangle - |-'_1 \otimes +'_2\rangle)/\sqrt{2}$$

(b) $\hat{a} = \hat{b} = \hat{z}$ $\quad ((S_1)_z \otimes (S_2)_z)|\Phi\rangle = (m_1\hbar \otimes m_2\hbar)|\Phi\rangle = (m_1\hbar \otimes -m_1\hbar)|\Phi\rangle$

They conclude that their spin-magnetic moments are opposite

$$((S_1)_x \otimes (S_2)_x)|\Phi\rangle \doteq \frac{\hbar}{2}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_1 \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_2\right)|\Phi\rangle = \frac{\hbar}{2}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\otimes\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\otimes\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)/\sqrt{2}$$

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma} \qquad S_x \doteq \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad S_y \doteq \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad S_z \doteq \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$((S_1)_z \otimes (S_2)_z)|\Phi\rangle \doteq \frac{\hbar}{2}\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}_1 \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}_2\right)|\Phi\rangle = \frac{\hbar}{2}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\otimes\begin{bmatrix} 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\otimes\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)/\sqrt{2}$$

?

$$-\frac{\hbar^2}{4}|\Phi\rangle$$

(c) Without loss of generality one can set $\hat{a} = \hat{z}$ and adjust $\hat{b}$ such that the angle between $\hat{a}$ and $\hat{b}$ remains θ, because from (a) $|\Phi\rangle$ is invariant under rotations.

$$|+\otimes +, \hat{b}\rangle = I \otimes \mathcal{D}^{(1/2)}(\theta, 0)\,|+\otimes +\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{bmatrix} |+\otimes +\rangle$$

$$= \left| + \otimes \begin{bmatrix} \cos\theta/2 \\ \sin\theta/2 \end{bmatrix} \right\rangle = \cos\theta/2\,|+\otimes +\rangle + \sin\theta/2\,|+\otimes -\rangle$$

$$a_{++} = \langle +\otimes[+,\hat{b}]|\Phi\rangle = \left(\cos\frac{\theta}{2}\langle +\otimes +| + \sin\frac{\theta}{2}\langle +\otimes -|\right)|\Phi\rangle$$

$$= \cos\left(\frac{\theta}{2}\right)\cdot 0 + \sin\left(\frac{\theta}{2}\right)\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}$$

(d) $P_{++} = |a_{++}|^2 = \frac{1}{2}\sin^2\frac{\theta}{2}$

$$P_{+-} = |a_{+-}|^2 = |\langle +\otimes -, \hat{b}|\Phi\rangle|^2 = \left|\left(-\sin\theta/2\,\langle +\otimes +| + \cos\theta/2\,\langle +\otimes -|\right)|\Phi\rangle\right|^2 = \frac{1}{2}\cos^2\frac{\theta}{2}$$

$$P_{-+} = |a_{-+}|^2 = |\langle -\otimes +, \hat{b}|\Phi\rangle|^2 = \left|\left(\cos\theta/2\,\langle -\otimes +| + \sin\theta/2\,\langle -\otimes -|\right)|\Phi\rangle\right|^2 = \frac{1}{2}\cos^2\frac{\theta}{2}$$

$$P_{--} = |a_{--}|^2 = |\langle -\otimes -, \hat{b}|\Phi\rangle|^2 = \left|\left(-\sin\theta/2\,\langle -\otimes +| + \cos\theta/2\,\langle -\otimes -|\right)|\Phi\rangle\right|^2 = \frac{1}{2}\sin^2\frac{\theta}{2}$$

$$E(\hat{a},\hat{b}) = \langle \varepsilon_a \varepsilon_b \rangle = \sum_{\varepsilon_a, \varepsilon_b} \varepsilon_a \varepsilon_b P_{\varepsilon_a \varepsilon_b} = ++\frac{1}{2}\sin^2\frac{\theta}{2} + -\frac{1}{2}\cos^2\frac{\theta}{2} - +\frac{1}{2}\cos^2\frac{\theta}{2} - -\frac{1}{2}\sin^2\frac{\theta}{2}$$

$$= \sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2} = -\cos\theta = -\hat{a}\cdot\hat{b}$$